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Power loss surface for a [pi] circuit with susceptive shunt elements

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POWER LOSS SURFACE FOR A π CIRCUIT WITH SUSCEPTIVE SHUNT ELEMENTS

by

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INTRODUCTION

It has been 75 years since Edison's Pearl Street generating station first went into operation. In this span of time electric power systems have increased tremendously in size and complexity. One of the problems that has been made vastly more difficult by this growth is the economic dispatching of the electrical load. This dispatching is more difficult because in the present day interconnected electric-power systems the power losses associated with the high voltage transmission lines is comparable to the power losses associated with the generating stations. Therefore the most economic allocation of the electric power from the generating stations in the system cannot be made unless the variation in transmission line losses with respect to station generation is also known.

In the past 15 years a great deal of work has been done in the way of developing computational methods for including transmission line losses in the economic dispatch problem. It was the study of this economic dispatch problem that aroused the author's interest in transmission line losses and led to the present investigation. However this investigation digresses from the problem which served as a stimulus in that it focuses attention on the losses of a particular circuit and not on the losses of an entire system. The basic interest in minimizing transmission line losses led directly to the π circuit with susceptive shunt branches because any results obtained from studying the π circuit would be directly applicable to the power system transmission line. At the same time the π circuit is not limited to representing transmission lines, hence the results can be interpreted in terms of any part of an electrical system that can be represented by the π circuit.

Specifically the purpose of this investigation was to develop the power loss surface for a π circuit with susceptive shunt branches and then to study the surface with respect to its minimum points.

In reviewing the literature relating to electrical losses the author was unable to find anything which was directed along the line of this investigation. From the literature it is apparent that previous investigators of electrical losses have directed their efforts into two general areas. The first area that attracted the interest of early investigators was the general problem of accurately calculating electrical losses. The second area of interest was the study of the most economic operation of power systems[1].

For example in reviewing the literature relating to transmission line losses the author found that the writers were concerned either with techniques for calculating line losses or with techniques for including the line losses in the economic dispatch problem. This is not surprising. The early investigators recognized that with the advent of the high voltage long distance transmission line an accurate determination of line losses would require the solution of the line as a distributed parameter circuit. Since the calculation of the line losses by this rigorous solution of the transmission line entailed a great deal of time and effort

[1] A representative list of references from these two areas of general interest is given in the Selected References.

the early investigators spent considerable time looking into mathematical short cuts, charts, nomographs and similar techniques to simplify the calculations.

The economic dispatch problem is of tremendous importance to those engaged in the distribution and sale of electric power. Hence it is not surprising to find the interest lies in minimizing the total system loss rather than the losses of a single element.

This investigator felt that the development of the power loss surface for a π circuit with particular attention to the minimum points on the surface would contribute some knowledge to the general problem of minimizing electrical losses.

THE INVESTIGATION

The purpose of the investigation was to develop and study the power loss surface for a π circuit with susceptive shunt elements. In particular the analysis was concerned with locating the points on the surface corresponding to minimum power loss. In addition to locating the minimum points the explicit relationships among the variables at these minimum points was also sought.

The development of the power loss surface was carried out in four steps. The first step was to develop the expression for the power loss P_o as a function of the five variables in the system, namely: the input voltage E_1; the output voltage E_2; the circuit resistance R; the circuit reactance X; and the output power P_2. Stated mathematically the first step in the development of the power loss surface was to formulate an expression of the form:

$$P_o = f(E_1, E_2, R, X, P_2). \qquad 1$$

The second step consisted of combining the variables into the dimensionless combinations: (E_2/E_1); (X/R); and (RP_2/E_1^2). The dimensionless combinations were arrived at by means of Buckingham's Pi theorem.

The third step involved rewriting Equation 1 in terms of the dimensionless variables (E_2/E_1), (X/R), and (RP_2/E_1^2). Thus:

$$P_o = g[(X/R), (E_2/E_1), (RP_2/E_1^2)]. \qquad 2$$

The fourth step comprised the study of the level surfaces generated by Equation 2 with (RP_2/E_1^2) as the parameter.

Development of the Power Loss Equation

The functional relationship defined in Equation 1 was derived on the basis of the schematic diagram shown in Figure 1. Figure 1 illustrates the conventional π circuit. The shunt susceptive branches are shown for completeness. The shunt susceptance does not enter into the loss equation and hence B is not a variable in the investigation. The positive sense of the terminal voltages and powers is defined in Figure 1. The expressions for the terminal powers P_1 and P_2 are:

$$P_1 = \frac{E_1^2 R}{Z^2} - \frac{E_1 E_2}{Z^2} [R \cos\delta - X \sin\delta]; \qquad 3$$

and

$$P_2 = \frac{E_2^2 R}{Z^2} - \frac{E_1 E_2}{Z^2} [R \cos\delta + X \sin\delta]. \qquad 4$$

The power loss of the circuit is:

$$P_o = P_1 + P_2 = \frac{R}{Z^2} [E_1^2 + E_2^2 - 2E_1 E_2 \cos\delta]. \qquad 5$$

From Equation 4 it is possible to obtain an expression for δ in terms of E_1, E_2, R, X, and P_2. The derivation is as follows. Rewrite Equation 4 as:

$$\frac{R}{Z} \cos\delta + \frac{X}{Z} \sin\delta = \frac{E_2^2 R - Z^2 P_2}{E_1 E_2 Z}. \qquad 6$$

Recognize that

$$\frac{R}{Z} = \cos\theta, \qquad 7$$

$$\frac{X}{Z} = \sin\theta, \qquad 8$$

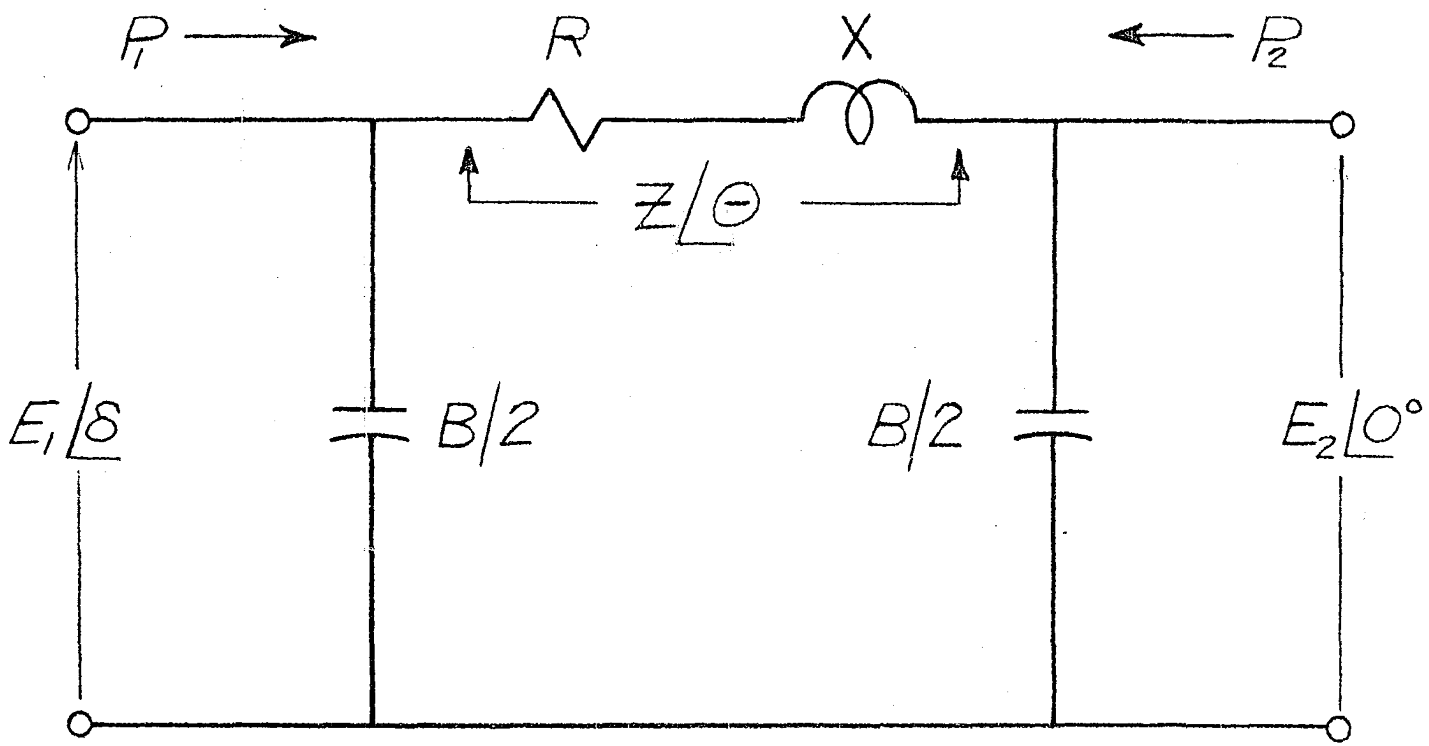

Figure 1. Schematic diagram of the π circuit

and

$$\cos\theta\cos\delta + \sin\theta\sin\delta = \cos(\theta - \delta), \qquad 9.$$

and hence Equation 6 can be written

$$\cos(\theta - \delta) = \frac{E_2{}^2R - Z^2P_2}{E_1E_2Z}. \qquad 10$$

From Equation 10

$$\theta - \delta = \cos^{-1}\frac{E_2{}^2R - Z^2P_2}{E_1E_2Z}. \qquad 11$$

For convenience let

$$\beta = \cos^{-1}\frac{E_2{}^2R - Z^2P_2}{E_1E_2Z}; \qquad 12$$

then

$$\delta = \theta - \beta; \qquad 13$$

and

$$\cos\delta = \cos(\theta - \beta) = \cos\theta\cos\beta + \sin\theta\sin\beta. \qquad 14$$

From the definition of β the expressions for $\cos\beta$ and $\sin\beta$ are readily obtained. Thus:

$$\cos\beta = \frac{E_2{}^2R - Z^2P_2}{E_1E_2Z}; \qquad 15$$

and

$$\sin\beta = \sqrt{1 - \cos^2\beta} = \frac{1}{E_1E_2Z}\sqrt{(E_1E_2Z)^2 - (E_2{}^2R - Z^2P_2)^2}. \qquad 16$$

The substitution of Equations 15 and 16 into Equation 14 gives:

$$\cos\delta = \frac{1}{E_1E_2Z^2}\left\{R(E_2{}^2R - Z^2P_2) + X\sqrt{(E_1E_2Z)^2 - (E_2{}^2R - Z^2P_2)^2}\right\}. \qquad 17$$

The desired expression for the line loss is obtained by substituting

Equation 17 into 5. The result is:

$$P_o = \frac{R}{Z^4}\left\{(E_1^2 + E_2^2)Z^2 + 2RP_2Z^2 - 2E_2^2R^2 - 2X\sqrt{(E_1E_2Z)^2 - (E_2^2R - Z^2P_2)^2}\right\}. \tag{18}$$

Equation 18 is the explicit expression for P_o as a function of E_1, E_2, R, X, and P_2 as symbolized by Equation 1.

Combining the Variables Into Dimensionless Combinations

The Buckingham Pi theorem was used to find the dimensionless combinations of the six variables E_1, E_2, R, X, P_2 and P_o. Thus for the variables involved in this problem the requirement that

$$P_o^aE_1^bE_2^cR^dX^eP_2^f = \text{numeric} \tag{19}$$

takes the form

$$(E^2Z^{-1})^aE^bE^cZ^dZ^e(E^2Z^{-1})^f = E^oZ^o. \tag{20}$$

Rewriting Equation 20 as

$$E^{2a+b+c+2f} \cdot Z^{-a+d+e-f} = E^oZ^o \tag{21}$$

leads directly to the relationships:

$$2a + b + c + 2f = 0 \tag{22}$$

$$-a + d + e - f = 0. \tag{23}$$

An inspection of the matrix of a system tells which unknowns can be solved for in terms of the others. The matrix for this system is

$$\begin{array}{cccccc} a & b & c & d & e & f \end{array}$$
$$\left\|\begin{array}{cccccc} 2 & 1 & 1 & 0 & 0 & 2 \\ -1 & 0 & 0 & 1 & 1 & -1 \end{array}\right\|.$$

The rank of this matrix is 2. Now the rank of the matrix of the coefficients of the unknowns to be solved for must equal the rank of the matrix of the system. For example it is not possible to solve for a and f in terms of b, c, d, and e, because the matrix of the coefficients a and f has a rank of 1. That is the rank of the matrix

$$\begin{Vmatrix} 2 & 2 \\ -1 & -1 \end{Vmatrix}$$

is 1. Thus the possible combinations of unknowns in terms of the remaining unknowns are:

I	a	and	b	in	terms	of	c, d, e, f
II	a	"	c	"	"	"	b, d, e, f
III	a	"	d	"	"	"	b, c, e, f
IV	a	"	e	"	"	"	b, c, d, f
V	b	"	d	"	"	"	a, c, e, f
VI	b	"	e	"	"	"	a, c, d, f
VII	b	"	f	"	"	"	a, c, d, e
VIII	c	"	d	"	"	"	a, b, e, f
IX	c	"	e	"	"	"	a, b, d, f
X	c	"	f	"	"	"	a, b, d, e
XI	d	"	f	"	"	"	a, b, c, e
XII	e	"	f	"	"	"	a, b, c, d.

Explicitly the relations I through XII are:

I $\quad a = d + e - f$

$\quad b = -c - 2d - 2e$

II $\quad a = d + e - f$

$\quad c = -b - 2d - 2e$

III $a = -0.5(b + c + 2f)$

$d = -0.5(b + c) - e$

IV $a = -0.5(b + c + 2f)$

$e = -0.5(b + c) - d$

V $b = -2a - 2f - c$

$d = a + f - e$

VI $b = - 2a - 2f - c$

$e = a + f - d$

VII $b = - c - 2d - 2e$

$f = d + e - a$

VIII $c = - 2a - 2f - b$

$d = a + f - e$

IX $c = - 2a - 2f - b$

$e = a + f - d$

X $c = - b - 2d - 2e$

$f = - a + d + e$

XI $d = -0.5(b + c) - e$

$f = -0.5(b + c + 2a)$

XII $e = -0.5(b + c) - d$

$f = -0.5(b + c + 2a)$.

The relationships I through XII lead directly to the twelve sets of π groups given in Table I.

The question of which set of π groups to select was done by a process of elimination. The elimination process was based on rewriting Equation 1 in terms of the dimensionless π groups π_1, π_2, π_3 and π_4. Thus

Table I

Dimensionless Combinations of the Six Variables

$E_1 - E_2 - R - X - P_2 - P_o$

Set Number	Buckingham Pi Groups			
	π_1	π_2	π_3	π_4
I	E_2/E_1	P_oR/E_1^2	P_oX/E_1^2	P_2/P_o
II	E_1/E_2	P_oR/E_2^2	P_oX/E_2^2	P_2/P_o
III	$E_1/\sqrt{P_oR}$	$E_2/\sqrt{P_oR}$	X/R	P_2/P_o
IV	$E_1/\sqrt{P_oX}$	$E_2/\sqrt{P_oX}$	R/X	P_2/P_o
V	P_oR/E_1^2	E_2/E_1	X/R	P_2R/E_1^2
VI	P_oX/E_1^2	E_2/E_1	R/X	P_2X/E_1^2
VII	P_o/P_2	E_2/E_1	P_2R/E_1^2	P_2X/E_1^2
VIII	P_oR/E_2^2	E_1/E_2	X/R	P_2R/E_2^2
IX	P_oX/E_2^2	E_1/E_2	R/X	P_2X/E_2^2
X	P_o/P_2	E_1/E_2	P_2R/E_2^2	P_2X/E_2^2
XI	P_o/P_2	$E_1/\sqrt{P_2R}$	$E_2/\sqrt{P_2R}$	X/R
XII	P_o/P_2	$E_1/\sqrt{P_2X}$	$E_2/\sqrt{P_2X}$	R/X

$$\pi_1 = h(\pi_2, \pi_3, \pi_4). \qquad 24$$

From Equation 24 it was evident that an explicit equation for P_o could be obtained if P_o appeared in only one π group. Hence the first four sets were eliminated.

The selection of which set of the remaining π groups would be most suitable reduced to a study of the original circuit with regard to the most likely independent variables. Viewing the circuit in terms of possible variables eliminated the resistance R in the sense that for R equal to zero no loss problem existed. Recognizing the physical impossibility of reducing R to zero made X the most logical variable in terms of the series impedance of the circuit. Therefore all remaining sets in which X appeared in more than one π group were eliminated. Explicitly sets VI, IX, and XII.

The second independent variable that was selected was P_2. It seemed reasonable to assume that the performance of the circuit at different power output levels would be of primary concern. Therefore P_2 should appear in only one π group and thus sets VII, X, and XI were eliminated.

Having selected X and P_2 as independent variables the selection of the third and final variable was reduced to making a choice between the terminal voltages E_1 and E_2. The choice of E_2 as the third independent variable was based on the feeling that in most applications of π circuits the input voltage E_1 would be insensitive to changes in the output voltage E_2. With E_2 as the third variable, set VIII was eliminated and this left V as the most desirable set.

Substituting the π groups of set V into Equation 24 gave the functional relationship:

$$P_o = \frac{E_1^2}{R} h \left[\frac{E_2}{E_1}, \frac{X}{R}, \frac{RP_2}{E_1^2}\right] . \qquad 25$$

Hence the next step in the development of the loss surface was to rewrite

Equation 18 in terms of the ratios (E_2/E_1), (X/R), and (RP_2/E_1^2) in order to obtain the explicit form of Equation 25.

Development of the Loss Equation in Terms of the Dimensionless Variables (E_2/E_1), (X/R) and (RP_2/E_1^2)

To find the power loss as a function of the dimensionless ratios (E_2/E_1), (X/R), and (RP_2/E_1^2) the quantity $E_1^2R^2$ was factored out of the bracketed term of Equation 18. Thus Equation 18 became:

$$P_o = \frac{E_1^2}{R\left[1 + \left(\frac{X}{R}\right)^2\right]^2}\left\{\left[1 + \left(\frac{E_2}{E_1}\right)^2\right]\cdot\left[1 + \left(\frac{X}{R}\right)^2\right] + \frac{2RP_2}{E_1^2}\left[1 + \left(\frac{X}{R}\right)^2\right] - 2\left(\frac{E_2}{E_1}\right)^2 - 2\sqrt{\left(\frac{X}{R}\right)^2\left(\frac{E_2}{E_1}\right)^2\left[1 + \left(\frac{X}{R}\right)^2\right] - \left(\frac{X}{R}\right)^2\left[\left(\frac{E_2}{E_1}\right)^2 - \frac{RP_2}{E_1^2}\left[1 + \left(\frac{X}{R}\right)^2\right]\right]^2}\right\} \qquad 26$$

In working with Equation 26 it was convenient to make the following simplifications in nomenclature: $U = (X/R)^2$; $V = (E_2/E_1)^2$; and $W = (RP_2/E_1^2)$. Thus Equation 26 simplified to:

$$P_o = \frac{E_1^2}{R(1+U)^2}\left\{(1+V)(1+U) + 2W(1+U) - 2V - 2\sqrt{UV(1+U)-U[V-W(1+U)]^2}\right\}. \qquad 27$$

Determining the level surfaces generated by Equation 27 with W treated as a parameter comprised the fourth and final step of the investigation.

Determination of the Power Loss Surface

When any one of the variables U, V, or W of Equation 27 is considered constant the resulting equation is that of a surface in three dimensional space. A family of surfaces is generated as the "constant" takes on various values. The power loss surface was developed on the basis that W would be the parameter that generated a family of surfaces. Hence for a given value of W the output power of the π circuit would be fixed and the resulting surface would give the power loss as a function of U and V. The selection of W as the parameter was based on the premise that studying the operation of the π circuit at fixed power outputs would be more realistic than studying the operation of the circuit at fixed voltage ratios V or fixed impedance ratios U.

With W considered constant Equation 27 took the form

$$P_o = F(U, V). \qquad 28$$

The nature of the surface described by Equation 28 was determined by cutting the surface with constant P_o, U, and V planes. The results of cutting the power loss surface with constant P_o, U, and V planes are given in the following three subsections.

U vs. V for constant values of P_o

Letting a represent a particular value of W and k represent the ratio R/E_1^2 Equation 27 was written in the form:

$$k(1+U)^2P_o = (1+V)(1+U) + 2a(1+U) - 2V - 2\sqrt{UV(1+U) - U[V-a(1+U)]^2}. \qquad 29$$

To remove the radical sign in Equation 29 the equation was rewritten so that the radical term appeared on one side of the equation and the remaining terms on the other side. The resulting expression was squared and then factored into the following expression:

$$(1+U)^2 \Big\{(kP_o)^2U^2 - 2kP_oUV + 2U[2a^2 + P_o{}^2k^2 - kP_o(1+2a)] + V^2 - 2V(1+2a-kP_o) + (1+2a-kP_o)^2\Big\} = 0. \qquad 30$$

With U a positive number by definition, the term inside the brackets of Equation 30 had to equal zero. Thus:

$$(kP_o)^2U^2 - 2kP_oUV + 2U[2a^2 + P_o{}^2k^2 - kP_o(1+2a)] + V^2 - 2V(1+2a-kP_o) + (1+2a-kP_o)^2 = 0. \qquad 31$$

For a constant value of P_o Equation 31 is a second degree equation in U and V. Explicitly it is of the form:

$$AU^2 + BUV + CV^2 + DU + EV + F = 0. \qquad 32$$

The nature of the U - V locus for constant values of P_o is determined by the value of $B^2 - 4AC$ in Equation 32. Equation 32 has as a locus a circle (if $A = C$, $B = 0$), an ellipse (if $B^2 - 4AC < 0$), a hyperbola (if $B^2 - 4AC > 0$), or a parabola (if $B^2 - 4AC = 0$). For Equation 32:

$$B^2 - 4AC = 4k^2P_o{}^2 - 4k^2P_o{}^2 = 0. \qquad 33$$

Therefore the U vs. V trace in a constant P_o plane was found to be a parabola.

The characteristics of the U - V parabola's were obtained by solving Equation 31 for: (a) V as a function of U; (b) U as a function of V; and (c) the angle of rotation which would eliminate the UV term. These solutions led to the following equations:

$$V = (1+2a-kP_o) + UkP_o \pm 2\sqrt{U[kP_o - (a-kP_o)^2]} \qquad 34$$

$$U = \frac{1}{k^2P_o^2}\left\{(VkP_o-a^2) + [P_ok - (a-kP_o)^2]\right.$$

$$\left. \pm 2\sqrt{(VkP_o-a^2)[kP_o - (a-kP_o)^2]}\right\} \qquad 35$$

$$\tan 2\phi = \frac{2\tan\phi}{1-\tan^2\phi} = \frac{B}{A-C} = \frac{2kP_o}{1-k^2P_o^2}. \qquad 36$$

Studying Equations 34 through 36 led to the following observations concerning the U - V parabolas:

1. The parabolas degenerate into straight lines at the values of P_o that make the term $[kP_o - (a-kP_o)^2]$ equal to zero. Hence these values of P_o represent the maximum and minimum values of the power loss in the circuit. Explicitly:

$$P_{o_{max}} = \frac{(1+2a) + \sqrt{1+4a}}{2k} \qquad 37$$

$$P_{o_{min}} = \frac{(1+2a) - \sqrt{1+4a}}{2k}. \qquad 38$$

Furthermore at these values of P_o the equations of the UV lines were found to be:

at $P_{o_{max}}$

$$V = \frac{U}{2}\left[1+2a + \sqrt{1+4a}\right] + \frac{1}{2}\left[1+2a - \sqrt{1+4a}\right] \qquad 39$$

at $P_{o_{min}}$

$$V = \frac{U}{2}\left[1+2a - \sqrt{1+4a}\right] + \frac{1}{2}\left[1+2a + \sqrt{1+4a}\right]. \qquad 40$$

2. At U equal to zero V is single valued and has the value

$(1+2a - kP_o)$. This point is where the parabola is tangent to the P_o - V co-ordinate plane. As the value of P_o decreases from its maximum to its minimum value this tangent point traces the straight line

$$V = 1 + 2a - kP_o \qquad 41$$

in the P_o - V co-ordinate plane.

3. The U - V parabola is also tangent to the line: $V = a^2/kP_o$. The U co-ordinate at this point of tangency is $[kP_o - (a-kP_o)^2]/k^2P_o^2$. As P_o passes from its maximum to its minimum value this tangent point traces a parabolic curve. The projection of this curve onto the U - V co-ordinate plane was found to be:

$$V^2 - (1+2a)V + a^2U + a^2 = 0 \qquad 42$$

or;

$$U = \frac{1}{a^2}[V - (V-a)^2]. \qquad 43$$

4. The principal axis of the parabola has a slope equal to kP_o. Therefore as P_o decreases from its maximum to its minimum value the principle axis of the parabola rotates clockwise.

These characteristics of the U - V parabolas are shown graphically in Figures 2 and 3.

<u>P_o vs. V for constant values of U</u>

For a constant value of U Equation 31 has as a locus an ellipse. This was established by letting <u>b</u> represent a particular value of U and then putting Equation 31 in the form:

$$k^2(b+1)^2P_o^2 + 2k(1-b)VP_o + V^2 - 2k(1+2a)(b+1)P_o - 2(1+2a)V + 4a^2b + (1+2a)^2 = 0. \qquad 44$$

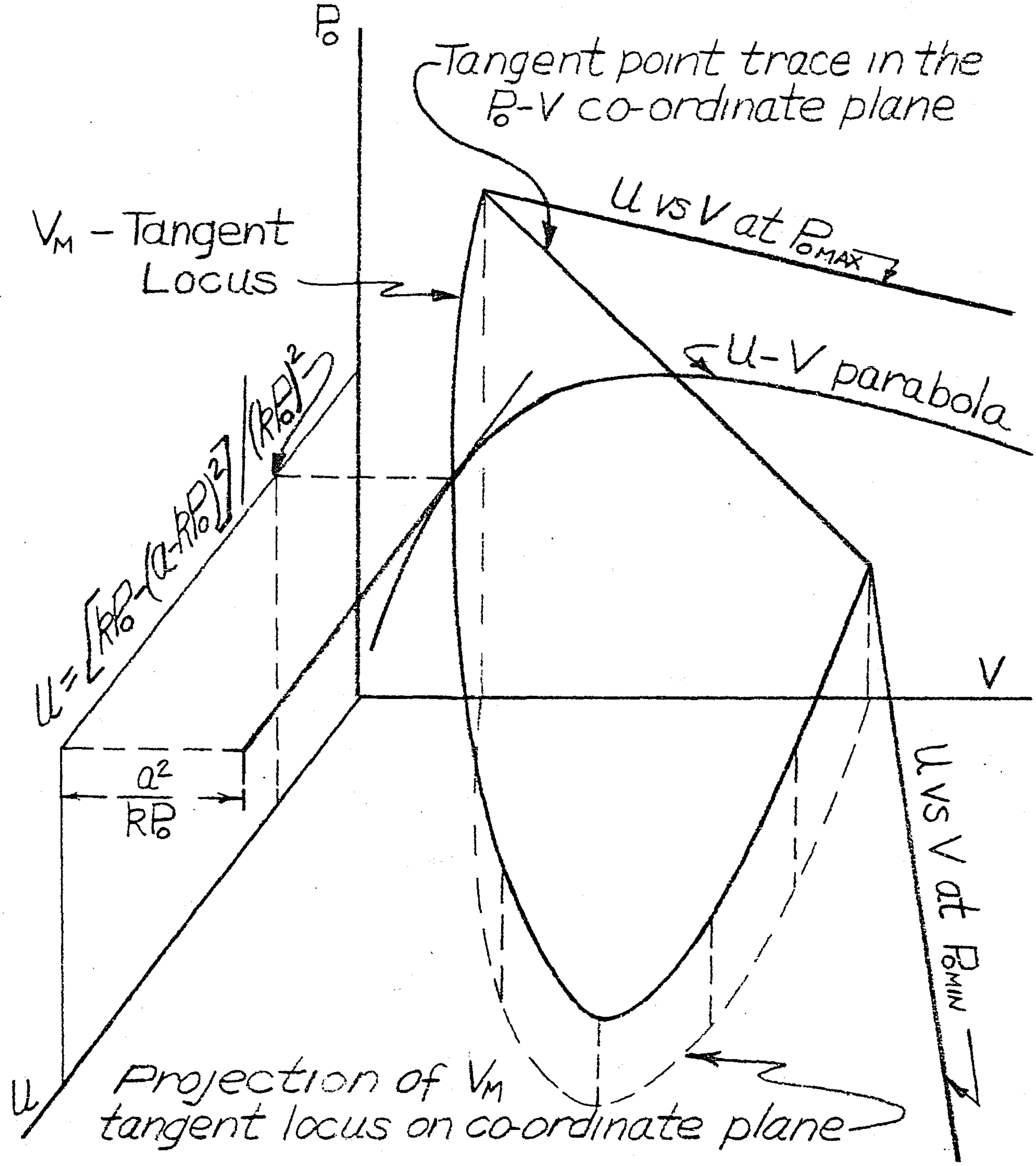

Figure 2. A three dimensional view illustrating the orientation of the U - V parabolas

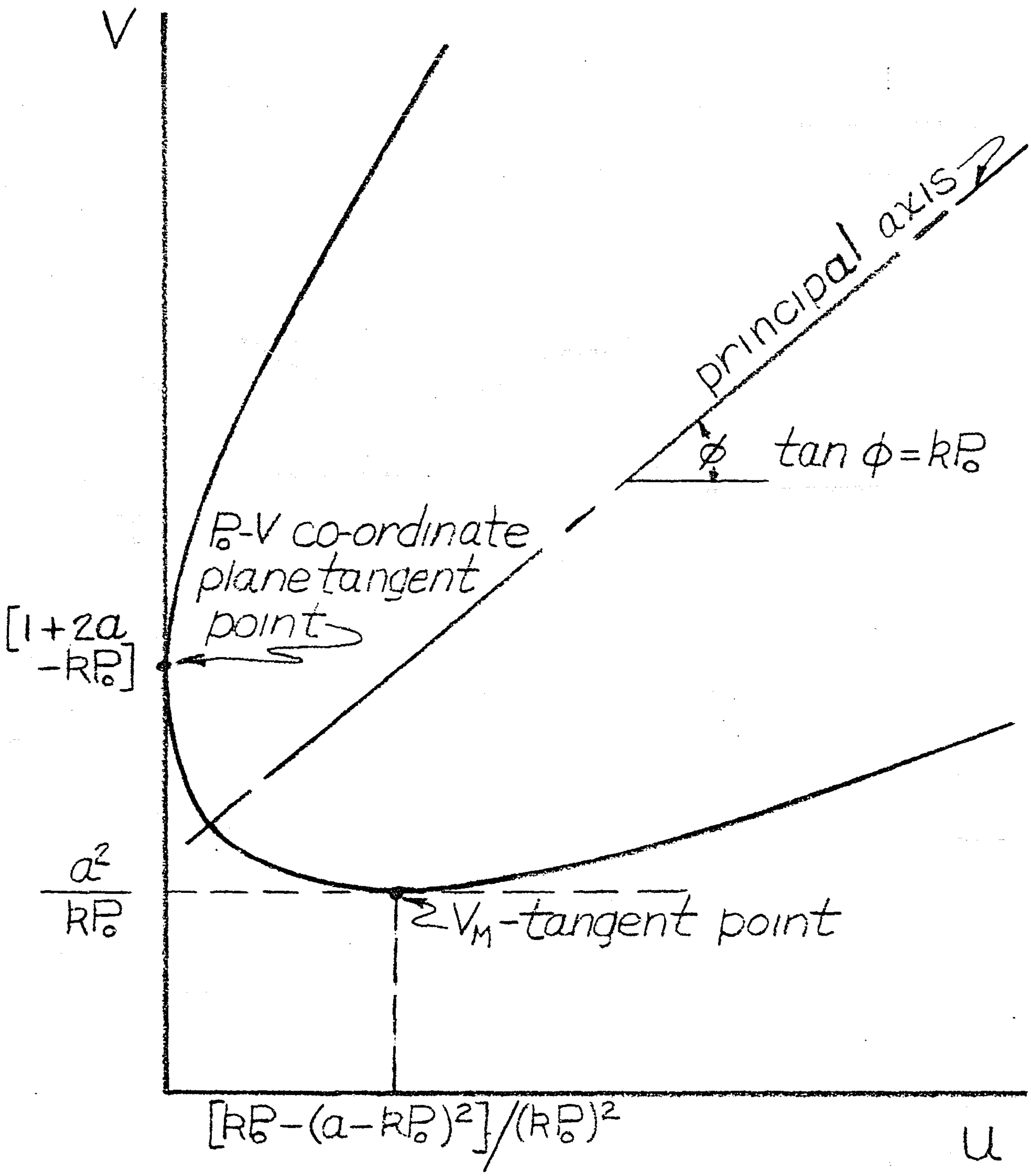

Figure 3. A two dimensional view illustrating the orientation of the U - V parabolas

Then recognizing Equation 44 as a second degree equation of the form:

$$AP_o^2 + BVP_o + CV^2 + DP_o + EV + F = 0; \qquad 45$$

the elliptical locus was established by noting that the characteristic $B^2 - 4AC$ was less than zero. Explicitly:

$$B^2 - 4AC = [2k(1-b)]^2 - 4k^2(b+1)^2 = -16\ bk^2. \qquad 46$$

The following characteristics of the $P_o - V$ ellipses were noted from a study of Equation 44.

1. The centers of the $P_o - V$ ellipses were found to lie along the straight line

$$V = \frac{1+2a}{2}\ U + \frac{1+2a}{2} \qquad 47$$

in the constant P_o plane $(1+2a)/2k$. Explicitly the centers of the ellipses lie at the co-ordinate points:

$$V = \frac{(1+2a)(b+1)}{2} \qquad 48$$

$$P_o = (1+2a)/2k. \qquad 49$$

2. At b equals zero the ellipse degenerates into the straight line $V = 1 + 2a - kP_o$. This line in the $P_o - V$ co-ordinate plane corresponds to the points where the U - V parabolas are tangent to the $P_o - V$ co-ordinate plane.

3. To eliminate the P_oV term in Equation 44 the $P_o - V$ co-ordinate axes would have to be rotated through an angle θ such that:

$$\tan 2\theta = \frac{2k(1-b)}{1-k^2(b+1)^2} \qquad 50$$

(A positive value of θ corresponds to a counter-clockwise rotation of the axes.)

From Equation 50 it was noted that at b equal to zero the tan θ was equal to k and at b equal to one the tan θ was equal to zero. Thus the value of θ at b equal to zero agreed with the slope of the straight line formed by the degenerate ellipse; and the value of θ at b equal to one corresponded to the intersection of the straight lines formed by the degenerate parabolas.

4. The points along the ellipse where dP_o/dV was equal to zero were found by differentiating Equarion 44 with respect to V and solving for dP_o/dV. Thus

$$\frac{dP_o}{dV} = \frac{(1+2a) - V - kP_o(1-b)}{k^2P_o{}^2(b+1)^2 + kV(1-b) - k(1+2a)(b+1)}. \qquad 51$$

For dP_o/dV to be equal to zero requires

$$(1+2a) - V - kP_o(1-b) = 0. \qquad 52$$

Solving Equation 52 for kP_o and substituting this value back into Equation 44 led to the following:

$$V = \frac{1}{2}(1+2a)(b+1) \pm \frac{1}{2}(b-1)\sqrt{1+4a}. \qquad 53$$

Since Equation 53 is equivalent to Equations 39 and 40 the slope along a P_o - V ellipse was found to be zero at the points where the U - V parabolas degenerated into straight lines.

5. The minimum and maximum values of V along the P_o - V ellipse were obtained by solving Equation 44 for kP_o. Thus:

$$kP_o = \frac{(1+2a)(b+1) - (1-b)V \pm 2\sqrt{b\left[V(1+2a)(b+1) - V^2 - a^2(b+1)^2\right]}}{(b+1)^2}. \qquad 54$$

The limiting values of V are the values of V that make the radical term in

Equation 54 zero. Explicitly:

$$V_{min} = \frac{b+1}{2}\left(1+2a - \sqrt{1+4a}\right) \qquad 55$$

$$V_{max} = \frac{b+1}{2}\left(1+2a + \sqrt{1+4a}\right). \qquad 56$$

These characteristics of the P_o - V ellipses are shown graphically in Figures 4 and 5.

P_o vs. U for constant V planes

Having cut the power loss surface with constant P_o planes and constant U planes it remained to investigate the P_o vs. U locus for constant values of V. Letting c represent a particular value of V, Equation 31 became:

$$k^2P_o{}^2(U+1)^2 - 2kP_o[U(c+1+2a) - c+1+2a] + 4a^2U + [c-(1+2a)]^2 = 0. \qquad 57$$

Solving Equation 57 for kP_o gave:

$$kP_o = \frac{U(c+1+2a)-c+1+2a \pm 2\sqrt{U\left\{(U+1)c-[a(U+1)-c]^2\right\}}}{(U+1)^2}. \qquad 58$$

From Equation 57 or Equation 58 it was apparent that the P_o - U locus could not be described in terms of circles, parabolas, hyperbolas, or ellipses. Never-the-less further insight into the nature of the loss surface was obtained by making this cut. The results of cutting the loss surface with constant P_o planes and constant U planes had established the maximum and minimum points on the surface as those points along the straight lines described by Equations 39 and 40. However from Equation 40 it was apparent that only the minimum points corresponding to V greater than $[1+2a + \sqrt{1+4a}]$ were known. For values of V between $[1+2a - \sqrt{1+4a}]$

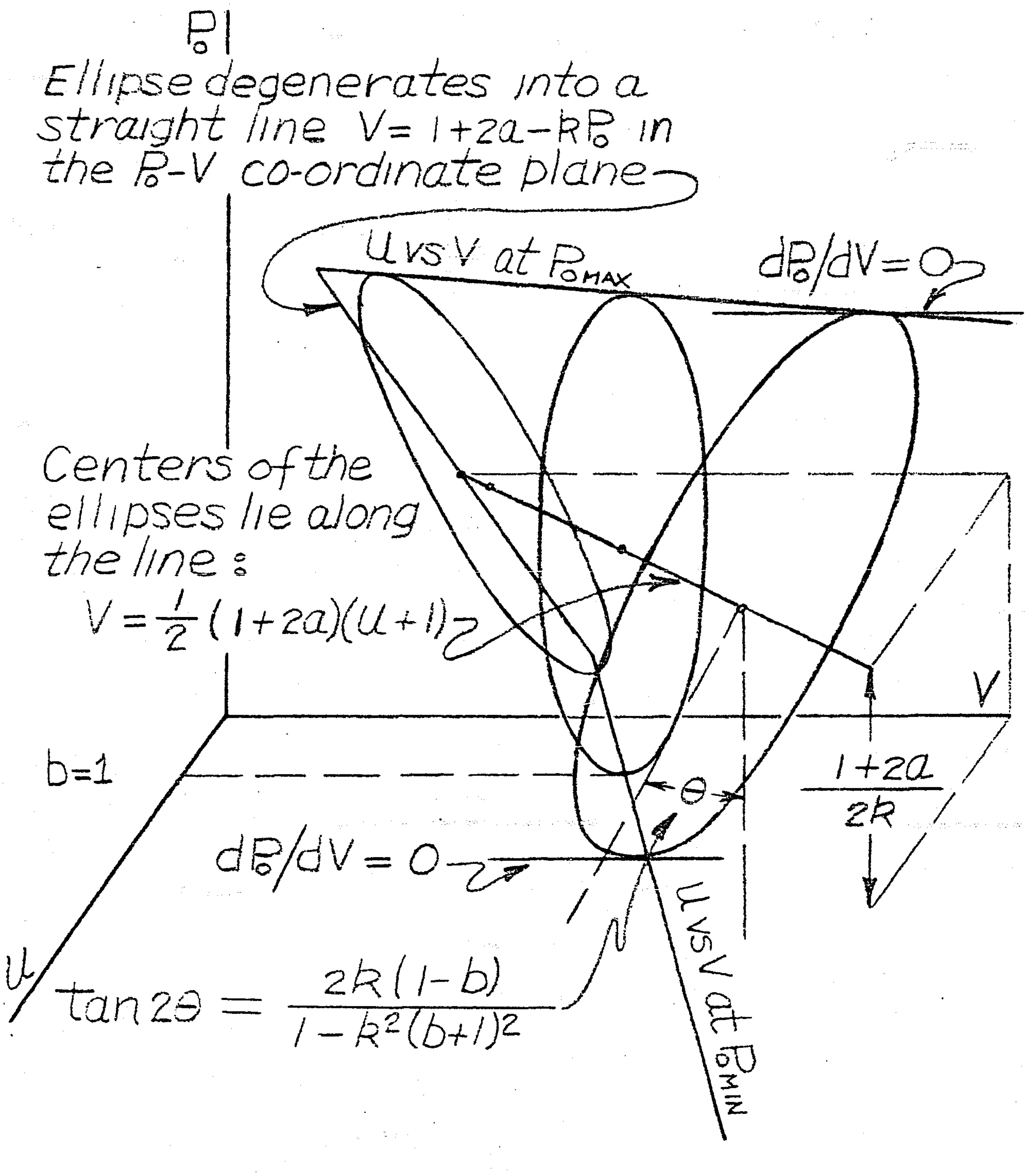

Figure 4. A three dimensional view illustrating the orientation of the P_o - V ellipses

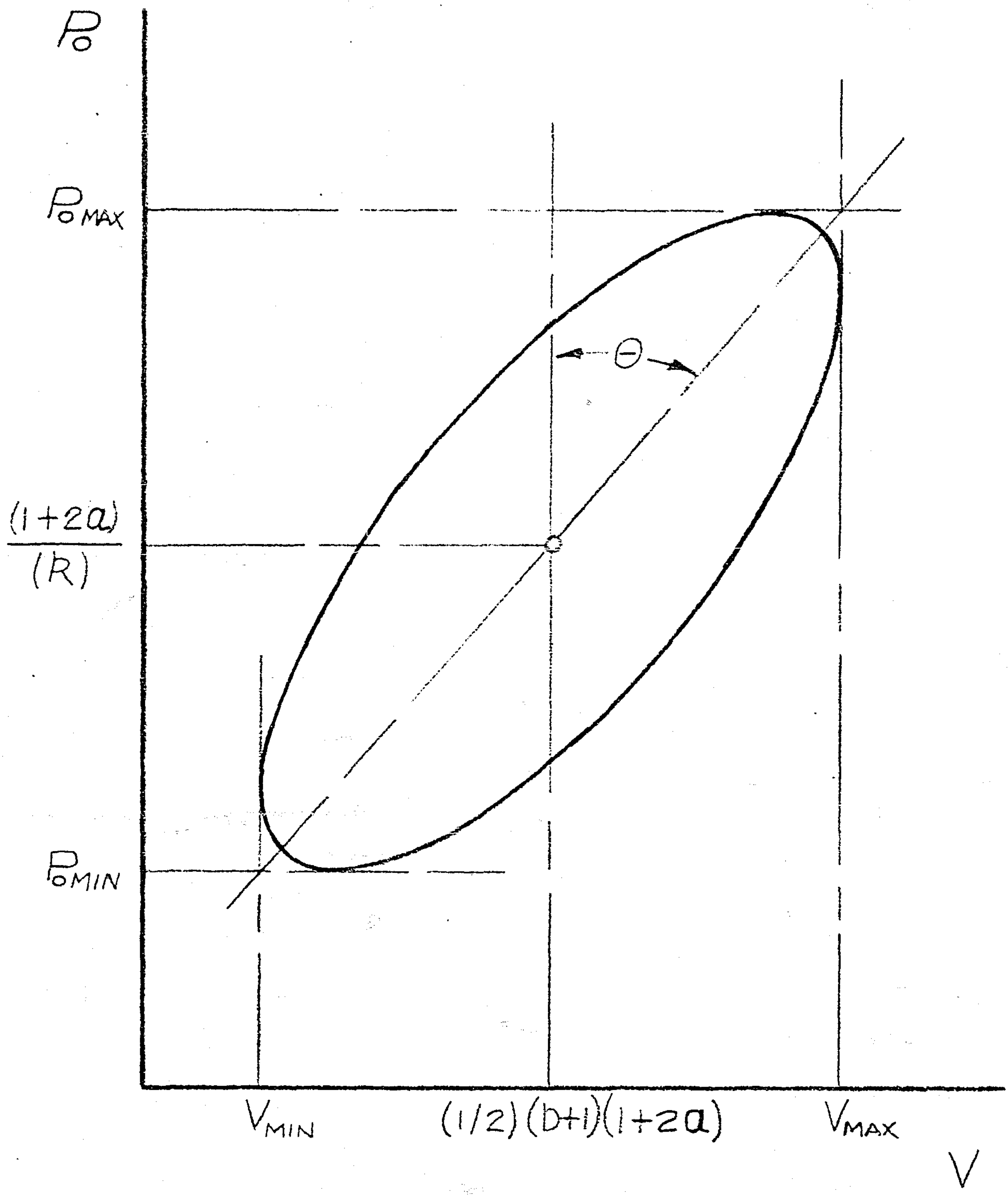

Figure 5. A two dimensional view illustrating the orientation of the P_o - V ellipses for b > 1

and $[1+2a + \sqrt{1+4a}]$ the locus of minimum points had not been established. To find this locus it was necessary to locate the points where dP_o/dU was equal to zero.

From Equation 57 the expression for dP_o/dU was found to be:

$$\frac{dP_o}{dU} = \frac{[kP_o(c+1+2a) - k^2P_o^2(U+1) - 2a^2]}{k\left\{kP_o(U+1)^2 - [U(c+1+2a)-c+1+2a]\right\}}. \qquad 59$$

Thus for dP_o/dU to equal zero it is necessary that:

$$kP_o(c+1+2a) - k^2P_o^2(U+1) - 2a^2 = 0. \qquad 60$$

To verify that the loci described by Equations 39 and 40 correspond to maximum and minimum points, Equation 60 was shown to be satisfied for these particular values of P_o, U, and c. Thus to show that dP_o/dU was zero along the line described by Equation 39, Equation 60 was shown to be satisfied for:

$$kP_o = \frac{1}{2}\,[(1+2a) + \sqrt{1+4a}] \qquad 61$$

and

$$c = \frac{1}{2}\,[U(1+2a + \sqrt{1+4a}) + (1+2a - \sqrt{1+4a})]. \qquad 62$$

To show that dP_o/dU was zero along the line described by Equation 40 the values:

$$kP_o = \frac{1}{2}\,[1+2a - \sqrt{1+4a}] \qquad 63$$

and

$$c = \frac{1}{2}\,[U(1+2a - \sqrt{1+4a}) + (1+2a + \sqrt{1+4a})] \qquad 64$$

were shown to satisfy Equation 60.

dP_o/dU was also found to be zero along the locus traced out by the V_M tangent point of the U - V parabolas. That is Equation 60 was also satisfied for:

$$U = \frac{[kP_o - (a-kP_o)^2]}{k^2P_o^2} \qquad 65$$

and

$$c = \frac{a^2}{kP_o}. \qquad 66$$

To determine whether or not the points along the V_M tangent point trace corresponded to a maximum, minimum, or inflection point on the P_o - U locus it was necessary to study the second derivative of P_o with respect to U along this curve. To study the second derivative Equation 59 was put in the form:

$$\frac{dP_o}{dU} = P_o' = \frac{F(P_o,U)}{G(P_o,U)}. \qquad 67$$

Thus

$$\frac{d^2P_o}{dU^2} = P_o'' = \frac{G(P_o,U)F'(P_o,U) - F(P_o,U)G'(P_o,U)}{[G(P_o,U)]^2}. \qquad 68$$

At P_o' equal to zero the expression for P_o'' is

$$P_o''\Big|_{P_o' = 0} = \frac{F'(P_o,U)}{G(P_o,U)}\Big|_{P_o' = 0} = \frac{-k^2P_o^2}{G(P_o,U)}. \qquad 69$$

Along the V_M tangent point trace Equation 69 becomes:

$$P_o'' = \frac{-a^6}{2c^3[(c-a)^2 - c]}. \qquad 70$$

From Equation 70 it was noted that the algebraic sign of P_o'' depends on the sign of the quantity $[(c-a)^2 - c]$ because c is positive by definition and a^6 is positive because it is a number raised to an even power. To study the nature of the quantity $[(c-a)^2 - c]$ the function H was created thus:

$$H = (c-a)^2 - c. \qquad 71$$

H was observed to have the following characteristics:

(a) $H = 0$ at $c = \frac{1}{2}[(1+2a) \pm \sqrt{1+4a}]$

(b) $\frac{dH}{dc} = -\sqrt{1+4a}$ at $c = \frac{1}{2}[(1+2a) - \sqrt{1+4a}]$

(c) $\frac{dH}{dc} = +\sqrt{1+4a}$ at $c = \frac{1}{2}[(1+2a) + \sqrt{1+4a}]$

(d) $\frac{dH}{dc} = 0$ at $c = \frac{1}{2}(1+2a)$

(e) $H = -\frac{1}{4}(1+4a)$ at $c = \frac{1}{2}(1+2a)$.

These characteristics of H are shown graphically in Figure 6.

With H established as a negative quantity in the interval $\frac{1}{2}(1+2a - \sqrt{1+4a}) < c < \frac{1}{2}(1+2a + \sqrt{1+4a})$ the second derivative P_o'' was found to be positive along the V_M tangent point locus and therefore these points correspond to minimum values of P_o.

The value of P_o along the V_M tangent point locus was found by substituting Equation 43 (with V replaced by c) into Equation 57. Thus along this curve:

$$P_{o_{min}} = \frac{E_1{}^2 a^2}{Rc}. \qquad 72$$

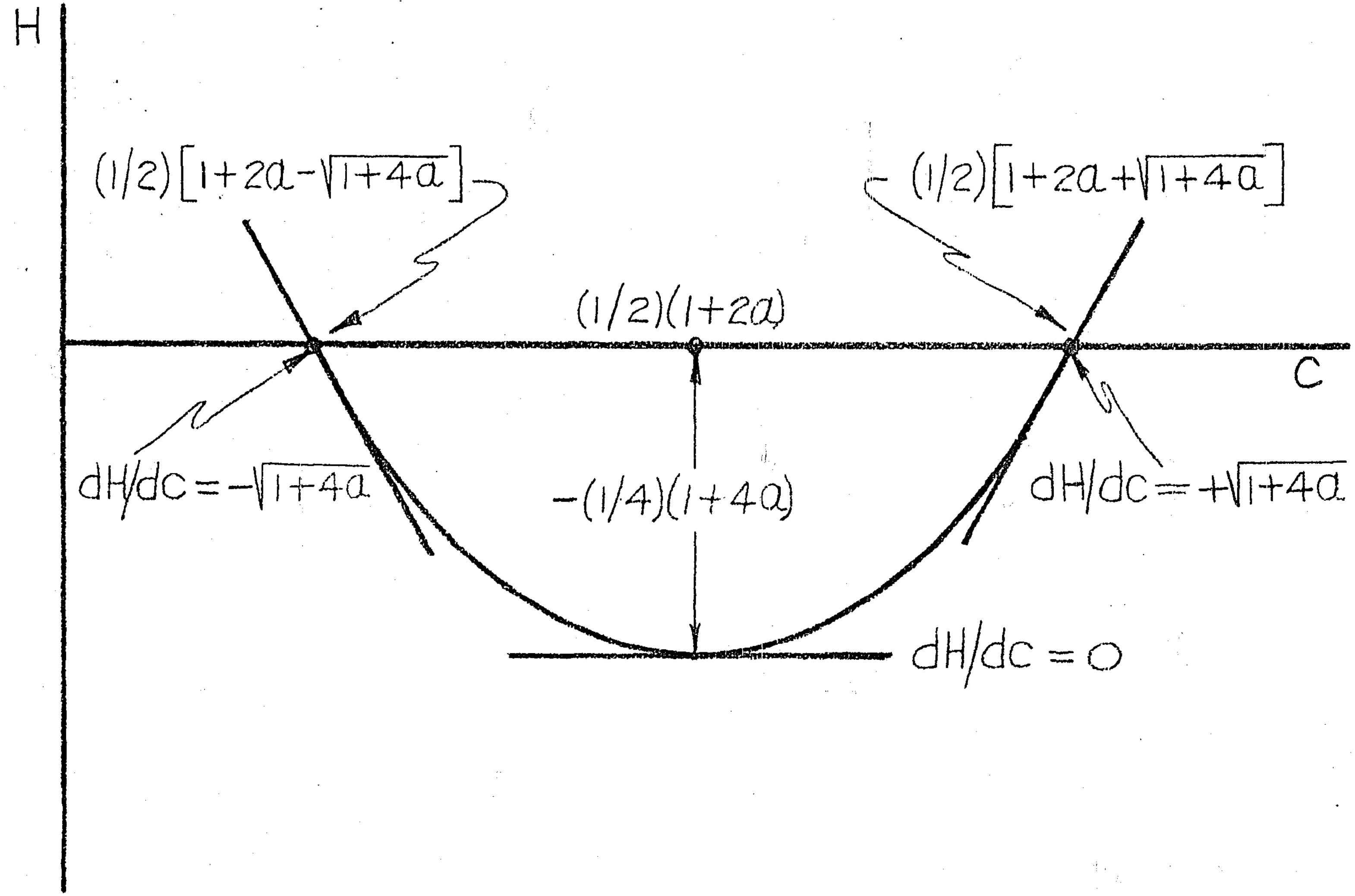

Figure 6. A graphical representation of H

Thus with regard to the maximum and minimum points the salient features of the P_o - U locus were established, namely:

1. For c greater than $1/2[1+2a - \sqrt{1+4a}]$ the maximum point on the P_o - U locus corresponded to the maximum point on the P_o - V locus which in turn corresponded to the line formed by the degenerate U - V parabola when P_o was equal to $P_{o_{max}}$.

2. For c greater than $1/2[1+2a + \sqrt{1+4a}]$ the minimum point on the P_o - U locus corresponded to the minimum point on the P_o - V locus which in turn corresponded to the line formed by the degenerate U - V parabola for P_o equal to $P_{o_{min}}$.

3. For values of c between $1/2[1+2a - \sqrt{1+4a}]$ and $1/2[1+2a + \sqrt{1+4a}]$ the minimum point on the P_o - U locus corresponded to the V_M tangent point of the U - V parabola, and has a value of $E_1{}^2a^2/Rc$.

In addition to the observations made regarding the maximum and minimum points of the P_o - U locus, further characteristics of the P_o - U locus were noted from Equation 58.

4. At U equal to zero Equation 58 reduces to $kP_o = 1+2a - c$. Thus the P_o - U loci and the U - V parabolas are tangent to the P_o - V co-ordinate plane at the same points.

5. The limiting values of U are obtained from the radical term in Equation 58. That is, solving the expression

$$(U+1)c - [a(U+1) - c]^2 = 0 \qquad 73$$

for U gives:

$$U = \frac{c}{a^2}\left[\frac{(1+2a) \pm \sqrt{1+4a}}{2}\right] - 1. \qquad 74$$

For $1/2[1+2a - \sqrt{1+4a}] \leq c \leq 1/2[1+2a + \sqrt{1+4a}]$ the minimum value of U is zero and the maximum value of U is:

$$u_{max} = \frac{c}{2a^2}\,[1+2a + 1+4a] - 1. \qquad 75$$

For all values of $c > 1/2[1+2a + \sqrt{1+4a}]$ the minimum and maximum values of U are given directly by Equation 74.

The fact that Equation 74 will not give the minimum value of U for $c < 1/2[1+2a + \sqrt{1+4a}]$ comes from the observation that for $c = 1/2[1+2a + \sqrt{1+4a}]$, U_{min} equals zero. For c less than this value the expression for U_{min} yields a negative number. By definition U is equal to or greater than zero, hence U_{min} is zero for $1/2[1+2a - \sqrt{1+4a}] \leq c \leq 1/2[1+2a + \sqrt{1+4a}]$.

DISCUSSION OF THE POWER LOSS SURFACE

The discussion of the power loss surface has been subdivided into five parts. The first part of the discussion deals with the limiting values of a and the nature of the surface at these values of a. The second phase of the discussion consists of noting the physical significance of the quantities k and a which appear in all the equations that describe the nature of the power loss surface. A table from which maximum and minimum values of P_o can be readily calculated comprises the third part of the discussion of results. The fourth section of the discussion is devoted to interpreting the P_o - V and P_o - U loci in terms of circuit operation. The fifth and final part of the discussion consists of numerical examples to illustrate the application of the results to an actual π circuit.

Power Loss Surface at the Limiting Values of a

The power loss surface was developed with the quantity $(RP_2/E_1^2 = a)$ as a parameter. Therefore a series of surfaces is created as a takes on various values. The first step in the study of the loss surface was to investigate the surfaces which correspond to the limiting values of a. The selected independent variables are X, E_2 and P_2 and therefore the range of a is determined by the range of P_2. The smallest value of P_2 is zero whereas the largest value is $E_1^2/4R$. P_2 is a negative number for power flows out of the output terminals and hence the parameter a lies between -1/4 and 0, thus:

$$-1/4 \leq a \leq 0. \qquad 76$$

For a equal to -1/4 the power loss surface collapses to a line in space. Thus at a equal to -1/4:

(a) $P_{o_{max}} = P_{o_{min}} = \frac{E_1^2}{4R}$; 77

(b) The U - V parabolas degenerate into the line

$$V = \frac{1}{4}(U+1); \quad 78$$

and

(c) The centers of the P_o - V ellipses are located at

$$V = \frac{1}{4}(b+1)$$

$$P_o = \frac{E_1^2}{4R}. \quad 79$$

Equation 77 expresses the well known fact that at maximum power transfer the losses are equal to the power output. Equations 78 and 79 show that the loss surface has been reduced to a straight line [$V = 1/4 (U+1)$] which is parallel to the U - V co-ordinate plane. The distance between the line and the plane is $E_1^2/4R$.

At a equal to zero the power loss surface takes on the following characteristics:

1. $P_{o_{max}} = \frac{E_1^2}{R}$. 80

2. $P_{o_{min}} = 0$. 81

3. At $P_{o_{max}}$ the U - V parabola degenerates into the line:

   $V = U$. 82

4. At $P_{o_{min}}$ the U - V parabola degenerates into the line:

   $V = 1$.

5. At U equal to zero the P_o - V ellipse degenerates into the line:

$$P_o = \frac{1}{k}(1-V). \quad 84$$

6. The centers of the P_o - V ellipses lie in the plane:

$$P_o = \frac{E_1^2}{2R} \quad 85$$

and describe the line

$$V = \frac{1}{2}(U+1) \quad 86$$

in this plane.

7. The expression for the U - V parabolas becomes:

$$V = UkP_o + (1 - kP_o) \pm 2\sqrt{UkP_o(1-kP_o)}. \quad 87$$

8. The V_M tangent point of the U - V parabolas lies in the P_o - U co-ordinate plane and traces out the curve:

$$P_o = \frac{1}{k} \cdot \frac{1}{(U+1)}. \quad 88$$

9. The expression for the P_o - V ellipses becomes:

$$P_o = \frac{1}{k}\left[\frac{Vb + (1+b-V) \pm 2\sqrt{Vb(1+b-V)}}{(b+1)^2}\right]. \quad 89$$

10. The limiting values of V along the P_o - V ellipses are:

$$V_{min} = 0$$

$$V_{max} = (b+1). \quad 90$$

11. The expression for the P_o - U loci becomes:

$$P_o = \frac{1}{k}\left[\frac{Uc + 1+U-c \pm 2\sqrt{Uc(1+U-c)}}{(U+1)^2}\right]. \quad 91$$

12. The limiting values of U along the P_o - U loci are:

$$U_{min} = 0$$

$$U_{max} = \infty. \quad 92$$

In interpreting the loss surface for a equal to zero it must be recognized that three different conditions may exist at the output terminals. That is, a will equal zero when:

1. The output terminals are open circuited;

2. The output terminals are short circuited;

or when

3. The output load is purely reactive.

When the output terminals are open circuited the power loss surface degenerates to the line V = 1 in the U - V co-ordinate plane. With no output power required, the minimum possible value of P_o is obviously zero. Zero loss implies zero current. This zero current condition will be satisfied if E_1 and E_2 are equal in magnitude and phase. Thus the line V = 1 in the U - V co-ordinate plane implies that these conditions exist at the terminals of the circuit.

When the output terminals are short circuited V will be identically zero and the loss surface reduces to the curve:

$$P_o = \frac{1}{k} \cdot \frac{1}{U+1}. \qquad 93$$

Replacing k by (R/E_1^2) and U by $(X/R)^2$ Equation 93 takes the familiar form:

$$P_o = \frac{E_1^2}{Z^2} R = I^2R. \qquad 94$$

Under this condition the power loss approaches zero as U approaches infinity.

When the π circuit is terminated in a purely reactive impedance the power loss will vary somewhere between zero and E_1^2/R depending on the value of the terminating reactance. When the terminating reactance is such that it resonates with the reactance of the series impedance of the circuit the power loss surface becomes the line U = V in the plane $P_o = 1/k$. This substantiates that in a series resonant circuit the power loss is E_1^2/R and the ratio of the voltages E_2 to E_1 equals the X/R ratio of the series impedance.

These observations concerning the nature of the power loss surface at the limiting values of a are of interest in that they show that results obtained from the power loss surface are in agreement with results which are readily obtained by other techniques. Therefore these observations serve to verify the validity of the general power loss surface.

Having looked at the limiting values of a and the nature of the loss surface at these values of a the next step in the analysis of the results was to note the physical significance of the quantities k and a.

Physical Significance of k and a

From Equation 80 the physical significance of k is apparent. Equation 80 states that the maximum possible value of $P_{o_{max}}$ is E_1^2/R and k is the reciprocal of this quantity. Thus:

$$\frac{1}{k} = \frac{E_1^2}{R} = P_{oo} \qquad 95$$

where P_{oo} represents the maximum possible value of $P_{o_{max}}$.

This observation with respect to k leads to the observation that the parameter a represents the ratio of the output power to the maximum possible loss in the circuit, that is

$$a = \frac{RP_2}{E_1^2} = \frac{P_2}{P_{oo}}. \qquad 96$$

Using the observation expressed in Equation 95 the expressions for $P_{o_{max}}$ and $P_{o_{min}}$ are written in the form:

$$\frac{P_{o_{max}}}{P_{oo}} = \frac{1 + 2a + \sqrt{1+4a}}{2} \qquad 97$$

and

$$\frac{P_{o_{min}}}{P_{oo}} = \frac{1 + 2a - \sqrt{1+4a}}{2}. \qquad 98$$

Numerical values of $P_{o_{max}}/P_{oo}$ and $P_{o_{min}}/P_{oo}$ for values of a between -1/4 and 0 are given in tabular form in the next section.

Tabulation of $P_{o_{max}}/P_{oo}$ and $P_{o_{min}}/P_{oo}$

Equation 98 does not lend itself to slide rule calculations because the quantity (1+2a) is approximately equal to $\sqrt{1+4a}$ for small values of a. [This is readily seen when the quantity $\sqrt{1+4a}$ is expanded by means of the binomial theorem. Thus: $\sqrt{1+4a} = 1+2a-2a^2+4a^3-10a^4+\cdots$.] For this reason numerical values of $P_{o_{max}}/P_{oo}$ and $P_{o_{min}}/P_{oo}$ to six significant figures have been worked out for increments of a equal to 0.005. These results are given in Table II. The values of $P_{o_{min}}/P_{oo}$ in Table II are applicable only if V is greater than $1/2[1+2a + \sqrt{1+4a}]$. For values of V less than this $P_{o_{min}}/P_{oo}$ can be calculated by means of Equation 72. Thus

$$\frac{P_{o_{min}}}{P_{oo}} = \frac{a^2}{c} \qquad 99$$

where c represents a value of V between

Table II

Values of $P_{o_{min}}/P_{oo}$ and $P_{o_{max}}/P_{oo}$ for a = 0.005

a	$P_{o_{min}}/P_{oo}$	$P_{o_{max}}/P_{oo}$	a	$P_{o_{min}}/P_{oo}$	$P_{o_{max}}/P_{oo}$
0.005	0.000025	0.989975	0.130	0.023590	0.716410
0.010	0.000102	0.979898	0.135	0.025883	0.704116
0.015	0.000232	0.969768	0.140	0.028337	0.691662
0.020	0.000417	0.959588	0.145	0.030963	0.679037
0.025	0.000658	0.949341	0.150	0.033772	0.666228
0.030	0.000958	0.939042	0.155	0.036779	0.653221
0.035	0.001319	0.928681	0.160	0.040000	0.640000
0.040	0.001742	0.918257	0.165	0.043452	0.626547
0.045	0.002231	0.907769	0.170	0.047157	0.612843
0.050	0.002786	0.897213	0.175	0.051138	0.598861
0.055	0.003412	0.886588	0.180	0.055425	0.584575
0.060	0.004110	0.875890	0.185	0.060049	0.569951
0.065	0.004884	0.865116	0.190	0.065051	0.554949
0.070	0.005736	0.854264	0.195	0.070479	0.539521
0.075	0.006670	0.843330	0.200	0.076393	0.523607
0.080	0.007689	0.832310	0.205	0.082868	0.507132
0.085	0.008798	0.821202	0.210	0.090000	0.490000
0.090	0.010000	0.810000	0.215	0.097917	0.472083
0.095	0.011299	0.798700	0.220	0.106795	0.453205
0.100	0.012701	0.787298	0.225	0.116886	0.433114
0.105	0.014211	0.775789	0.230	0.128578	0.411421
0.110	0.015834	0.764165	0.235	0.142525	0.387474
0.115	0.017516	0.752423	0.240	0.160000	0.360000
0.120	0.019445	0.740555	0.245	0.184289	0.325711
0.125	0.021447	0.728553	0.250	0.250000	0.250000

$$\frac{1}{2}\left[1+2a - \sqrt{1+4a}\right] \text{ and } \frac{1}{2}\left[1+2a + \sqrt{1+4a}\right].$$

A linear-interpolation for values of a between the tabulated values will give sufficiently accurate results. The largest error would occur for a midway between 0.245 and 0.250, or for a equal to 0.2475. For this value of a the per cent error [based on the true value of $(P_{o_{min}}/P_{oo})$] is 2.244. Thus:

$$\% \epsilon = \frac{\left[\frac{P_{o_{min}}}{P_{oo}}\right]_{interpolated} - \left[\frac{P_{o_{min}}}{P_{oo}}\right]_{calculated}}{\left[\frac{P_{o_{min}}}{P_{oo}}\right]_{calculated}} \cdot 100 \qquad 100$$

For a equal to 0.2475:

$$\left[\frac{P_{o_{min}}}{P_{oo}}\right]_{calculated} = \frac{1}{2}\left[1-0.495 - \sqrt{1-0.99}\right] = 0.2025 \qquad 101$$

and;

$$\left[\frac{P_{o_{min}}}{P_{oo}}\right]_{interpolated} = 0.184289 + \frac{1}{2}(0.250000 - 0.184289)$$

$$= 0.217144. \qquad 102$$

Therefore the per cent error is:

$$\% \epsilon = \frac{(0.217144 - 0.202500)100}{0.2025} = 2.244. \qquad 103$$

The interpolation error decreases for smaller values of a. For example the per cent error for a midway between 0.220 and 0.225 is:

$$\left[\frac{P_{o_{min}}}{P_{oo}}\right]_{calculated} = \frac{1}{2}\left[1-0.445 - \sqrt{1-0.89}\right] = 0.111669 \qquad 104$$

$$\left[\frac{P_{o_{min}}}{P_{oo}}\right]_{interpolated} = 0.106795 + \frac{1}{2}\left[0.116886 - 0.106795\right]$$

$$= 0.111840 \qquad 105$$

$$\% \epsilon = \frac{0.111840 - 0.111669}{0.111669} \times 100 = 0.1531\%. \qquad 106$$

Having constructed and evaluated a table from which $P_{o_{min}}/P_{oo}$ and $P_{o_{max}}/P_{oo}$ could be readily obtained the next step in the evaluation of the power loss surface was the interpretation of the P_o - V and P_o - U loci in terms of circuit operation.

Interpretation of P_o - V and P_o - U Loci

The P_o - V loci correspond to operating the π circuit with U and a constant. Hence these loci give the power loss of the circuit as a function of the terminal voltage ratio squared. It is apparent from the development of the P_o - V loci that P_o is a multiple valued function of V; that is, for each value of V, P_o has two values.

A physical interpretation of this comes most readily from Equation 4. By writing Equation 4 in the form:

$$P_2 = \frac{E_2 R}{Z^2} - \frac{E_1 E_2}{Z} \cos (\theta - \delta) \qquad 107$$

it is apparent that for given values of P_2, E_1, E_2 and R, and X (and hence U, V, and W) the equation can be satisfied for two values of δ. Since

$$\cos (\delta - \theta) = \cos (\theta - \delta) \qquad 108$$

Equation 107 will be satisfied when the magnitude of $(\delta - \theta)$ equals that of $(\theta - \delta)$. Let Υ equal the magnitude of $(\delta - \theta)$ then Equation 107 will hold for δ equal to $(\theta - \Upsilon)$ or $(\theta + \Upsilon)$. For each value of δ there will be a unique value of current and therefore a unique value of power loss.

These observations concerning the multiple valued nature of P_o are shown graphically in Figures 7 and 8. Figure 7 illustrates the two possible values of δ for a single value of P_2 and Figure 8 shows the two values

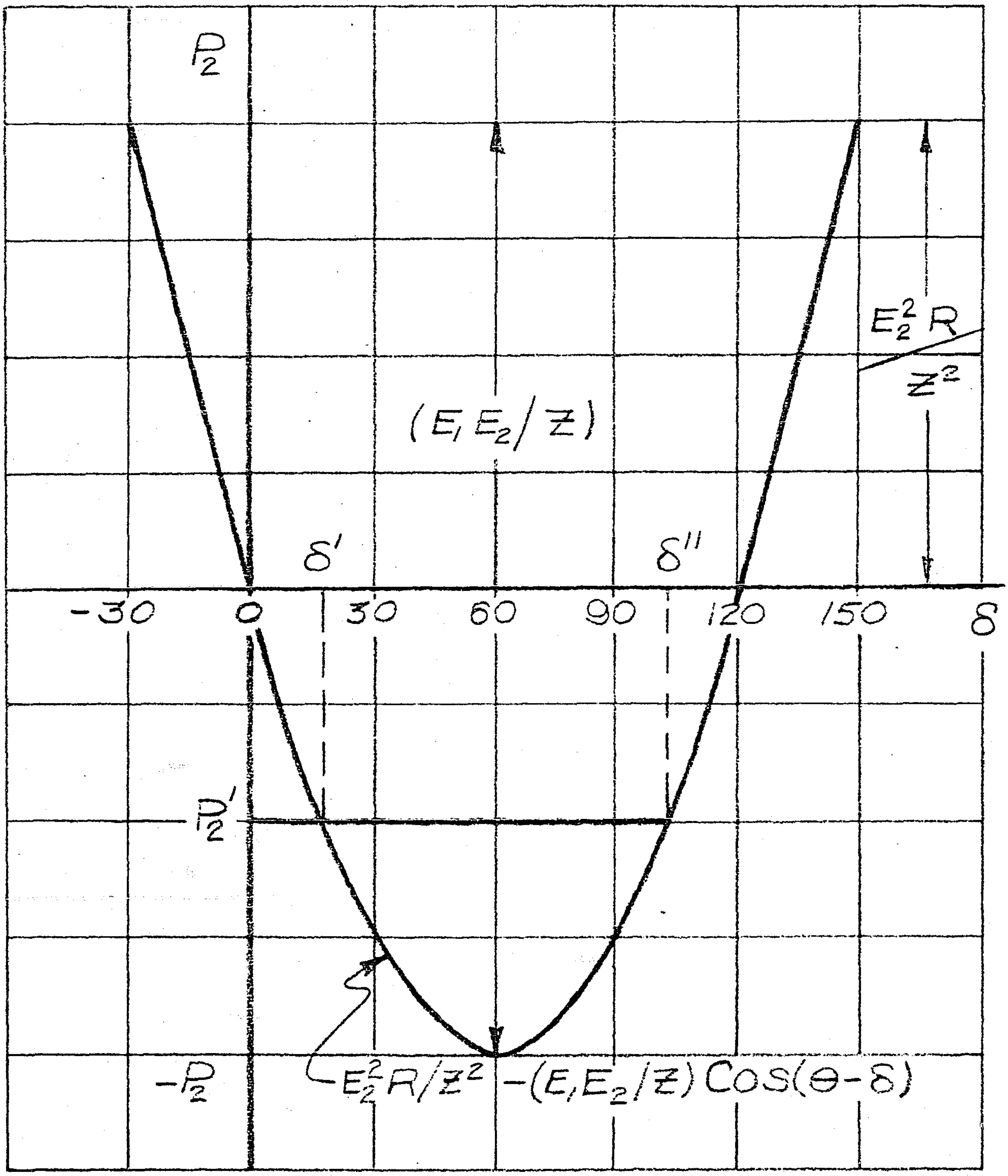

Figure 7. A graphical plot of the receiving end power showing that in general two values of δ correspond to a single value of P_2

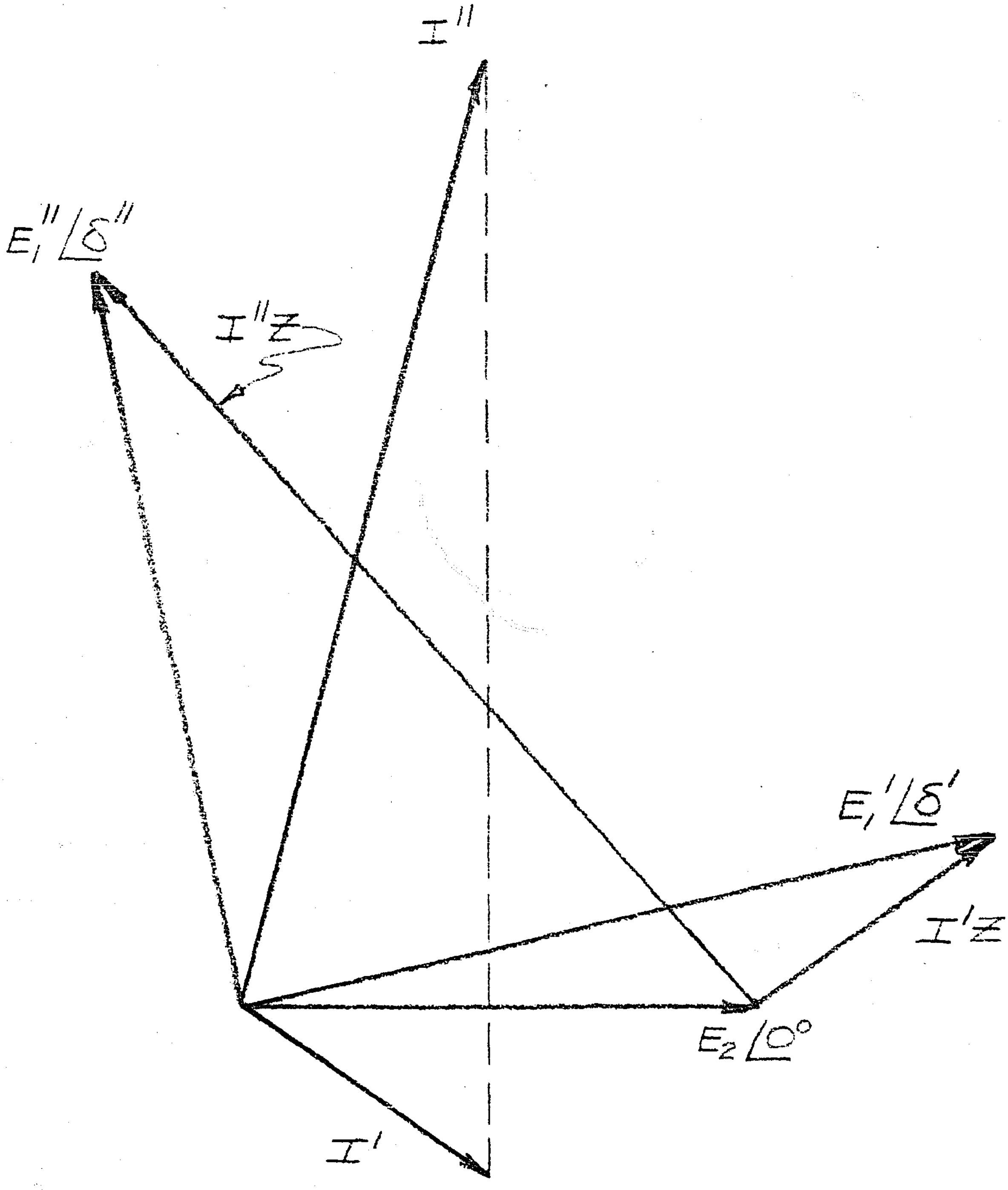

Figure 8. A vector diagram showing the two values of line current that correspond to the two values of δ

of current that correspond to the two values of δ.

The π circuit will be in stable operation when the angle δ has its smallest value. Therefore from an operating standpoint only the lower portion of the P_o - V locus is of interest. The limiting values of V on the locus give the range over which V can vary and still satisfy the specified operating conditions. The expressions for the limiting values of V are given by Equations 55 (minimum value) and 56 (maximum value). The value of V for which P_o is minimum is given directly by Equation 40.

As an illustration of the use of a P_o - V locus consider a π circuit where the resistance R, reactance X, input voltage E_1, and output power P_2 have been specified. Under these conditions U and a are constant and variations in V correspond to variations in the output voltage E_2. Hence a P_o - V locus represents the power loss in the circuit as a function of the output voltage E_2.

The limiting values of V become the limiting values of E_2 and these values of E_2 are calculated directly by means of Equations 55 and 56. The value of E_2 for which P_o is minimum comes directly from the value of V which corresponds to the minimum value of P_o. Thus Equation 40 gives the value of E_2 which minimizes the power loss in the circuit.

The interpretation of the P_o - U loci in terms of circuit operation is similar to the above discussion of the P_o - V loci. The P_o - U loci correspond to constant values of V and a and hence U becomes the independent variable. The multiple valued nature of P_o is apparent from Equation 58 and the interpretation of this double-valued characteristic in terms of circuit operation is identical with that given for the P_o - V loci.

The limiting values of U on the P_o - U locus give the range over which U can vary and still satisfy the specified operating conditions. The limiting values of U will depend on the value of V for which the P_o - U locus pertains. Equation 74 with the minus sign will give the minimum value of U for values of V greater than $1/2[(1+2a) + \sqrt{1+4a}]$. For all other permissible values of V, the minimum value of U is zero. The maximum value of U is always given by Equation 75.

The value of U corresponding to minimum power loss is given either by Equation 40 or by Equation 43. Equation 40 is applicable when V is greater than $1/2[1+2a + \sqrt{1+4a}]$ and Equation 43 applies when V is between $1/2[1+2a - \sqrt{1+4a}]$ and $1/2[1+2a + \sqrt{1+4a}]$.

As an illustration of when a P_o - U locus would be of interest consider a π circuit where the resistance R, input voltage E_1, output voltage E_2, and output power P_2 have been specified. Thus V and a are constant and variations in U correspond to variations in the reactance X. Thus the P_o - U locus gives the power loss in the circuit as a function of the reactance X. The limiting values of reactance, plus the reactance necessary to minimize the power loss, are directly calcuable from the appropriate equations which are in terms of U.

Numerical examples are given in the following section to illustrate the application of the results and to demonstrate numerically the observations that have been made with regard to the power loss surface.

Numerical Examples

The following numerical examples are given to illustrate the application of the results obtained from the development of the power loss surface. The examples are worked on a per unit basis to facilitate the calculations. Since the shunt susceptance of the π circuit did not enter into any of the power loss equations it is ignored entirely in the examples.

Example 1. As a first example consider a circuit where the following per unit quantities have been specified:

$E_1 = 1.0$; $R = 0.03$; and $P_2 = -3.0$.

With these quantities specified the development of the power loss surface will yield the maximum and minimum possible power loss. To find these values of P_o it is necessary to calculate P_{oo} and a, thus:

$$P_{oo} = \frac{E_1{}^2}{R} = \frac{100}{3} = 33\ \frac{1}{3} \qquad 109$$

and

$$a = \frac{P_2}{P_{oo}} = -0.09. \qquad 110$$

From Table II

$$\frac{P_{o_{max}}}{P_{oo}} = 0.81 \qquad 111$$

and

$$\frac{P_{o_{min}}}{P_{oo}} = 0.01. \qquad 112$$

Therefore the values of $P_{o_{max}}$ and $P_{o_{min}}$ become:

$$P_{o_{max}} = 0.81 \times \frac{100}{3} = 27 \qquad 113$$

and

$$P_{o_{min}} = 0.01 \times \frac{100}{3} = \frac{1}{3}. \qquad 114$$

From Equation 114 it is possible to determine the maximum possible efficiency that the circuit can operate at for these particular values of E_1, R, and P_2. Thus:

$$\% = \frac{\text{output}}{\text{input}} \times 100 = \frac{300}{3\frac{1}{3}} = 90\%. \qquad 115$$

Example 1 demonstrates that the minimum possible loss has been fixed by the specification of E_1, R, and P_2. The question naturally arises as to what values of E_2 and X will result in the circuit actually operating at this minimum power loss. Example 2 is given to show how the pertinent values of reactance can be obtained if E_2 is specified and Example 3 shows the calculations involving E_2 if the reactance has been specified.

Example 2. If in addition to specifying E_1, R, and P_2 the output voltage E_2 is also specified the results of the power loss surface will give the critical values of reactance. As an example suppose that E_2 is specified as 1.10. Before the desired values of U can be ascertained it is necessary to determine V and note whether V lies between $1/2[1+2a - \sqrt{1+4a}]$ and $1/2[1+2a + \sqrt{1+4a}]$ or whether V is greater than $1/2[1+2a + \sqrt{1+4a}]$.

For E_2 equal to 1.10 the value of V is:

$$V = \left[\frac{E_2}{E_1}\right]^2 = 1.21. \qquad 116$$

The critical values of V are:

$$V_{min} = \frac{1}{2}\left[1+2a - \sqrt{1+4a}\right] = 0.01 \qquad 117$$

and

$$V_{max} = \frac{1}{2}\left[1+2a + \sqrt{1+4a}\right] = 0.81. \qquad 118$$

Hence for E_2 equal to 1.10 the value of V is greater than V_{max} and therefore the limiting values of U can be obtained from Equation 74. Specifically:

$$U_{min} = \frac{c}{a^2}\left[\frac{1+2a - \sqrt{1+4a}}{2}\right] - 1 \qquad 119$$

$$U_{min} = \frac{1.21}{0.0081}\,[0.01] - 1 = 0.4938 \qquad 120$$

$$U_{max} = \frac{c}{a^2}\left[\frac{1+2a + \sqrt{1+4a}}{2}\right] - 1 \qquad 121$$

$$U_{max} = \frac{1.21}{0.0081}\,[0.81] - 1 = 120. \qquad 122$$

The limiting values of reactance can be obtained from Equations 120 and 122. The minimum value of reactance is:

$$X_{min} = R\sqrt{U_{min}} = 0.03\sqrt{0.4938} = 0.02108; \qquad 123$$

and the maximum value of reactance is:

$$X_{max} = R\sqrt{U_{max}} = 0.03\sqrt{120} = 0.3286. \qquad 124$$

The value of U which will give minimum power loss can be obtained by solving Equation 40 for U, thus:

$$U^* = \frac{V - \frac{1}{2}[1+2a + \sqrt{1+4a}]}{\frac{1}{2}[1+2a - \sqrt{1+4a}]} \qquad 125$$

where U^* represents the value of U at minimum power loss. Numerically:

$$U^* = \frac{1.21 - 0.81}{0.01} = 40. \qquad 126$$

The corresponding value of reactance is:

$$X^* = R\sqrt{U^*} = 0.03\sqrt{40} = 0.1897. \qquad 127$$

Hence for these specified values of E_1, E_2, R and P_2 the per unit reactance can be varied from 0.02108 to 0.3286 and for minimum power loss the per unit reactance must be 0.1897.

To illustrate the calculations when V is between $1/2[1+2a - \sqrt{1+4a}]$ and $1/2[1+2a + \sqrt{1+4a}]$ assume that E_2 is to be 0.80. V now equals 0.64 and the calculations involving U and $P_{o_{min}}$ become:

$$U_{min} = 0 \qquad 128$$

$$U_{max} = \frac{0.64}{0.0081}[0.81] - 1 = 63 \qquad 129$$

$$U^* = \frac{1}{a^2}[V - (V-a)^2] \qquad 130$$

$$U^* = \frac{1}{0.0081}[0.64 - 0.5329] = 13.22 \qquad 131$$

$$P_{o_{min}} = P_{oo}\left[\frac{a^2}{V}\right] = \frac{100}{3}\frac{0.0081}{0.64} = 0.4219. \qquad 132$$

The corresponding values of reactance are:

$$X_{min} = 0 \qquad 133$$

$$X_{max} = 0.03\sqrt{63} = 0.2381 \qquad 134$$

and

$$X^* = 0.03\sqrt{13.22} = 0.1091 \qquad 135$$

The next example demonstrates the calculations necessary for finding the pertinent values of E_2 when E_1, R, X, and P_2 have been specified.

Example 3. Let the numerical value of X be 0.24 then:

$$U = \left[\frac{X}{R}\right]^2 = \left[\frac{0.24}{0.03}\right]^2 = 64. \qquad 136$$

The limiting values of V come directly from Equations 55 and 56. Thus:

$$V_{min} = (b+1)\ \frac{1+2a - \sqrt{1+4a}}{2} \qquad 137$$

and

$$V_{max} = (b+1)\ \frac{1+2a + \sqrt{1+4a}}{2}\ . \qquad 138$$

Numerically:

$$V_{min} = (64+1)(0.01) = 0.65 \qquad 139$$

and

$$V_{max} = (64+1)(0.81) = 52.65. \qquad 140$$

The limiting values of E_2 can be obtained from the results given in Equations 138 and 140. Thus:

$$E_{2_{min}} = E_1\sqrt{V_{min}} = 1.0\sqrt{0.65} = 0.8062 \qquad 141$$

$$E_{2_{max}} = E_1\sqrt{V_{max}} = 1.0\sqrt{52.65} = 7.256. \qquad 142$$

The value of V corresponding to minimum power loss can be obtained from Equation 40:

$$V^* = U\ \frac{1+2a - \sqrt{1+4a}}{2} + \frac{1+2a + \sqrt{1+4a}}{2} \qquad 143$$

$$V^* = 64(0.01) + 0.81 = 1.45. \qquad 144$$

Therefore the value of E_2 corresponding to minimum loss becomes:

$$E_2^* = E_1\sqrt{V^*} = 1.0\sqrt{1.45} = 1.2041. \qquad 145$$

Thus with the per unit reactance set at 0.24 and with the restrictions that E_1 equals 1.0; R equals 0.03; and P_2 equals -3.0 the output voltage E_2 can be varied from 0.8062 to 7.256. Furthermore minimum power loss occurs when the output voltage is 1.204.

These three examples have shown how the results obtained from the development of the power loss surface can be applied to a π circuit when the input voltage E_1, resistance R and output power P_2 are specified. If the input voltage and output voltage exchange roles (that is, the output voltage is fixed and the input voltage is variable) the results of the power loss surface can still be applied. That the results of the power loss surface development can still be used is not immediately obvious because the parameters a and k become variables. Hence for changes in E_1 different power loss surfaces are generated and thus the simplicity of moving on a given surface has been lost.

The applicability of the already derived relationships for the case when E_2 is fixed and E_1 is variable was found by solving for the conditions of minimum power loss by another method and correlating the results with the loss surface. With E_2, P_2, R, and X specified, minimum power loss will occur when the current in the series impedance of the π circuit is in phase with the output voltage E_2. Thus the input voltage corresponding to minimum power loss is:

$$E_1^* = \sqrt{\left[E_2 + \frac{|P_2|R}{E_2}\right]^2 + \left[\frac{|P_2|X}{E_2}\right]^2} . \qquad 146$$

Removing the magnitude signs from P_2 and noting that:

$$Z^2 = R^2 + X^2 = R^2(1+U); \qquad 147$$

Equation 146 can be put in the form:

$$E_1^* = \sqrt{\frac{(E_2{}^2 - RP_2)^2 + UP_2{}^2R^2}{E_2{}^2}} . \qquad 148$$

At this value of E_1 the value of V becomes:

$$V^* = \left[\frac{E_2}{E_1^*}\right]^2 = \frac{E_2{}^4}{(E_2{}^2 - RP_2)^2 + UR^2P_2{}^2}. \qquad 149$$

Solving Equation 149 for U gives:

$$U = \frac{E_2{}^2E_1{}^{*2} - (E_2{}^2 - RP_2)^2}{R^2P_2{}^2}. \qquad 150$$

Multiplying Equation 150 by the ratio $(E_1{}^*/E_1{}^*)^4$ gives:

$$U = \frac{1}{a^{*2}}\,[V^* - (V^* - a^*)^2]. \qquad 151$$

The correlation with the previously derived results is now evident because this is the value of U corresponding to minimum power loss when V lies between $1/2(1+2a - \sqrt{1+4a})$ and $1/2(1+2a + \sqrt{1+4a})$. Thus the locus traced out by the V_M tangent point corresponds to the minimum possible loss when E_2, P_2, R, and X are specified and E_1 is the variable.

The minimum power loss for this operating condition is directly calcuable as:

$$P_{o_{min}} = \left[\frac{P_2}{E_2}\right]^2 R. \qquad 152$$

Equation 152 is in agreement with the power loss surface result as can be seen from the following:

$$\left.\begin{aligned} P_{o_{min}} &= \left[\frac{P_2}{E_2}\right]^2 R \cdot \frac{E_1{}^2}{R} \cdot \frac{R}{E_1{}^2} \cdot \frac{E_1{}^2}{E_1{}^2} \\ P_{o_{min}} &= \frac{R^2P_2{}^2}{E_1{}^4} \cdot \frac{E_1{}^2}{R} \cdot \frac{E_1}{E_2}^2 \\ P_{o_{min}} &= (a^2)(P_{oo})(\tfrac{1}{V}). \end{aligned}\right\} \qquad 153$$

The limiting values of E_1 are easily determined. The minimum value of E_1 is fixed by the maximum power transfer theorem, thus:

$$E_{1_{min}} = \sqrt{|4RP_2|}; \qquad 154$$

and the maximum possible value of E_1 is infinity.

The following example is given to show numerically the calculations involved for this particular operating condition.

Example 4. To correlate this example with some of the previous numerical results the specifications for the operation of the circuit are taken from Example 3. Thus: $R = 0.03$; $X = 0.24$; $P_2 = -3.0$; and $E_2^2 = 1.45$.

To find the value of E_1 which will give the minimum loss in the circuit Equation 43 is solved for E_1. The result is of course the same as Equation 148. Hence:

$$E_1^* = \sqrt{\frac{(E_2^2 - RP_2)^2 + UR^2P_2^2}{E_2^2}}; \qquad 155$$

and numerically:

$$E_1^* = \sqrt{\frac{(1.45+0.09)^2 + 64(0.0009)(9)}{1.45}}$$

$$E_1^* = \sqrt{1.9931} = 1.412. \qquad 156$$

The minimum value of P_o can be obtained either from Equation 72 or Equation 152. Thus:

$$P_{o_{min}} = \frac{9 \times 0.03}{1.45} = 0.1862. \qquad 157$$

The minimum possible value of E_1 is:

$$E_{1_{min}} = \sqrt{(4)(0.03)(3)} = \sqrt{0.36} = 0.6. \qquad 158$$

Therefore for this example the per unit input voltage can be increased from 0.6 to infinity and still satisfy the output requirements that E_2 = 1.2041 and P_2 = -3.0. For minimum loss in the circuit the per unit input voltage should be 1.412. The minimum value of P_o corresponding to this voltage is 0.1862.

It is interesting to compare this minimum loss with the minimum possible if E_1 is specified as 1.412 and E_2 is variable. For E_1 equal to 1.412 the parameters P_{oo} and $\underline{a}$ become:

$$P_{oo} = \frac{E_1{}^2}{R} = \frac{1.9913}{0.03} = 66.377 \qquad 159$$

$$a = \frac{P_2}{P_{oo}} = \frac{-3.0}{66.377} = 0.0452. \qquad 160$$

From Table II:

$$\frac{P_{o_{min}}}{P_{oo}} = 0.002231 + \frac{2}{50}\,[0.002786 - 0.002231]$$

$$\frac{P_{o_{min}}}{P_{oo}} = 0.002231 + 0.000022 = 0.002253. \qquad 161$$

Hence:

$$P_{o_{min}} = (66.377)(0.002253) = 0.1495. \qquad 162$$

The value of E_2 necessary to give this minimum power loss is:

$$V^* = 64(0.002253) + 0.907347 = 1.05154; \qquad 163$$

and therefore:

$$E_2{}^* = 1.412\sqrt{1.05154} = 1.4479. \qquad 164$$

Comparing the results given in Equations 157 and 162 shows numerically that the minimum loss is less if E_1 is specified and E_2 is allowed to vary. For this particular example the ratio of the minimums is:

$$\frac{P_{o_{min}}[E_2 \text{ variable}]}{P_{o_{min}}[E_1 \text{ variable}]} = \frac{0.1495}{0.1862} = 0.8029. \qquad 165$$

These numerical examples have been given with the idea that they demonstrate the application of the relationships that came out of the development of the power loss surface. A summary of the conclusions that can be made from the development of the surface is given in the next section.

SUMMARY OF RESULTS

The results obtained from the development of the power loss surface can be summarized as follows:

1. First of all the maximum and minimum points on the surface were clearly evident and therefore it was possible to obtain explicit relationships among the variables at these extreme points. These relationships are summarized below.

A. Maximum power loss relationships. The maximum value of P_o is:

$$P_{o_{max}} = P_{oo}\left[\frac{1+2a + \sqrt{1+4a}}{2}\right].$$

The relationship between U and V at $P_{o_{max}}$ is:

$$V = U\left[\frac{1+2a + \sqrt{1+4a}}{2}\right] + \left[\frac{1+2a - \sqrt{1+4a}}{2}\right].$$

B. Minimum power loss relationships. When V is greater than $1/2[1+2a + \sqrt{1+4a}]$ the minimum value of P_o is:

$$P_{o_{min}} = P_{oo}\left[\frac{1+2a - \sqrt{1+4a}}{2}\right].$$

The relationship between U and V at this value of $P_{o_{min}}$ is:

$$V = U\left[\frac{1+2a - \sqrt{1+4a}}{2}\right] + \left[\frac{1+2a + \sqrt{1+4a}}{2}\right].$$

When V lies between $1/2[1+2a - \sqrt{1+4a}]$ and $1/2[1+2a + \sqrt{1+4a}]$ the minimum value of P_o is:

$$P_{oo} = P_{oo} \cdot \frac{a^2}{V};$$

and the relationship between U and V is:

$$U = \frac{1}{a^2} [V - (V-a)^2].$$

With these explicit relationships the possibility of operating the circuit at its minimum loss point can be readily determined whenever restrictions are placed on some of the variables.

2. The development of the loss surface pointed out the possibility of operating the circuit at a relative minimum loss rather than at an absolute minimum loss. From the power loss surface it is evident that the circuit can only operate at a relative minimum whenever $1/2[1+2a - \sqrt{1+4a}] < V < 1/2[1+2a + \sqrt{1+4a}]$. For V equal to or greater than $1/2[1+2a + \sqrt{1+4a}]$ it is possible to operate the circuit at its absolute minimum loss. There are two possibilities that can force the circuit to operate at a relative minimum loss. First, with E_1, R, P_2 and E_2 specified the adjustment of X can only bring the circuit to a relative minimum if V lies within the range specified above. The second situation which only permits operation at a relative minimum is the case where E_2, R, X, and P_2 have been specified. In this case a variation of E_1 can only bring the circuit to a relative minimum point on the loss surface.

3. The absolute minimum loss is determined by the input voltage E_1, the resistance R, and the output power P_2. The relative minimum is determined by the output voltage E_2, the resistance R, and the output power P_2. Thus:

$$P_{o_{min}}(\text{absolute}) = \frac{E_1^2}{2R}\left[1 + \frac{RP_2}{E_1^2} - \sqrt{1 + \frac{4RP_2}{E_1^2}}\right];$$

and

$$P_{o_{min}}(\text{relative}) = \frac{RP_2{}^2}{E_2{}^2}.$$

4. The development of the power loss surface led directly to the P_o - V and P_o - U loci. These loci are of value in that they show how the power loss varies when either U or V is fixed. When U is fixed at some value b and V is variable the power loss as a function of V is:

$$P_o = \frac{1}{k}\left\{\frac{(1+2a)(b+1) - (1-b)V \pm 2\sqrt{b[V(1+2a)(b+1) - V^2-a^2(b+1)^2]}}{(b+1)^2}\right\}.$$

The value of V corresponding to the minimum value of P_o is:

$$V^* = b\left[\frac{1+2a - \sqrt{1+4a}}{2}\right] + \left[\frac{1+2a + \sqrt{1+4a}}{2}\right].$$

The limiting values of V are:

$$V_{min} = \frac{b+1}{2}(1+2a - \sqrt{1+4a})$$

$$V_{max} = \frac{b+1}{2}(1+2a + \sqrt{1+4a}).$$

When V is fixed at some value c and U is variable the power loss as a function of V is:

$$P_o = \frac{1}{k}\left\{\frac{U(c+1+2a) - c+1+2a \pm 2\sqrt{U\{(U+1)c - [a(U+1)-c]^2\}}}{(U+1)^2}\right\}.$$

The value of U corresponding to the minimum value of P_o depends on c. Thus:

For $1/2[1+2a - \sqrt{1+4a}] < c < 1/2[1+2a + \sqrt{1+4a}]$

$$U^* = \frac{1}{a^2}[c - (c-a)^2].$$

For $c \geq 1/2[1+2a + \sqrt{1+4a}]$

$$U^* = \frac{2c - [1+2a + \sqrt{1+4a}]}{[1+2a - \sqrt{1+4a}]}.$$

The limiting values of U also depend on c. Hence:

For $1/2[1+2a - \sqrt{1+4a}] \leq c \leq 1/2[1+2a + \sqrt{1+4a}]$

$$U_{min} = 0$$

$$U_{max} = \frac{c}{2a^2}[1+2a + \sqrt{1+4a}] - 1.$$

For $c > 1/2[1+2a + \sqrt{1+4a}]$

$$U_{min} = \frac{c}{2a^2}[1+2a - \sqrt{1+4a}] - 1$$

$$U_{max} = \frac{c}{2a^2}[1+2a + \sqrt{1+4a}] - 1.$$

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